Journal of Engineering Mathematics **27**: 189–207, 1993.

The connection between Ekman and Stewartson layers for a rotating disk

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Received 11 March 1992; accepted in revised form 24 June 1992

Abstract. When a disk of finite radius and the surrounding medium rotate coaxially with slightly different angular velocities, a so-called Stewartson layer exists at the edge of the disk. The properties of this layer outside the boundary layer of the disk have been given in a previous publication. In the present paper it is shown how the radial flow of the Ekman boundary layer turns into the axial flow of the Stewartson layer. This happens in a region of which both the radial and axial dimensions are $O(E^{1/2})$, where E is the Ekman number.

1. Introduction

This paper is a continuation of a previous paper "The Stewartson layer of a rotating disk of finite radius" by the author [1]. Stewartson [2] considered the shear layer existing between two coaxial rotating planes of which the center disks rotate with slightly different angular velocities. In [1] the shear layer at the edge ($r = 1$) of a rotating disk, placed in a medium which itself rotates with an angular velocity slightly different from that of the disk has been investigated. This shear layer reduces the axial velocity due to the influence of the boundary (Ekman) layer at the disk from $O(E^{1/2})$ for $r < 1$ to $O(E^{3/2})$ for $r > 1$, while in the shear layer itself the axial velocity is $O(E^{1/6})$. Here E denotes the Ekman number defined by $\nu/\Omega a^2$ with ν the kinematical viscosity coefficient, Ω the angular velocity of the disk and a the radius of the disk. The angular velocity of the medium is $(1 - \varepsilon)\Omega$.

There are more configurations for which such shear layers occur. In [3] Stewartson considered the case of two spheres again rotating with slightly different angular velocities. Moore and Saffman [4], [5] showed that shear layers also occur at the edge of a rising body in a rotating flow. In [6] they gave a very complete analysis of various cases of shear layers. The occurrence of shear layers in all these cases is due to the Taylor-Proudman theorem mentioning that outside the boundary layer the flow is independent of the axial coordinate when there is no viscosity, see Greenspan [7]. With viscosity the flow changes only at a distance $O(E^{-1})$, see [6] and [1].

In the present paper the connection between the Ekman layer along the disk and the Stewartson layer perpendicular to the disk is investigated. It was shown in [6] and used in [1] that the Ekman layer is governed by the classical boundary layer equations until a distance $O(E^{1/2})$ from the edge of the disk. Only within this distance the velocity derivatives in radial direction become of the same order as those in the direction normal to the disk. Although the Stewartson layer has a width $O(E^{1/3})$, all its fluid comes from the part of the boundary layer which lies within $O(E^{1/2})$ from the edge (Fig. 1). Therefore the relevant scaled coordinates in the region connecting the Ekman and Stewartson layers are $\tilde{r}$ and $\tilde{z}$, where

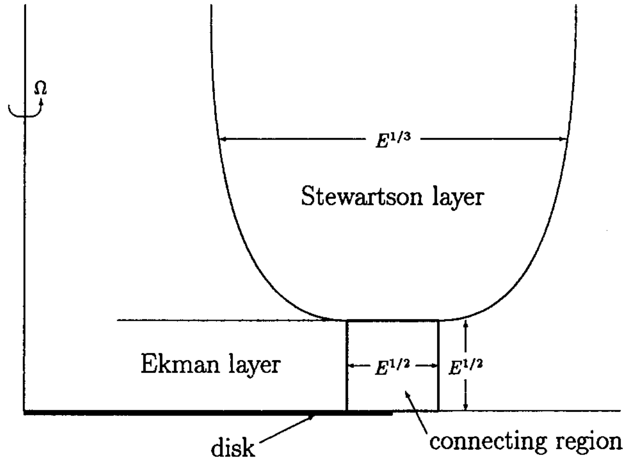

Fig. 1. The general configuration.

$\tilde{r} = (r-1)\mathrm{E}^{-1/2}$, $\tilde{z} = z\mathrm{E}^{-1/2}$

with r, z the dimensionless physical coordinates in radial and axial directions.

If the difference between the angular velocities of disk and medium is not small, that is, if $\varepsilon > O(\mathrm{E}^{1/6})$, see [1], then there exists at the edge a double-deck structure of size $O(\mathrm{E}^{3/7})$ in radial direction. F.T. Smith [8] considered a jet negotiating a trailing edge. Due to the fact that there is no external mainstream, a double-deck structure arises instead of a triple-deck. The double-deck arising in the case of a rotating disk in a quiescent medium has been calculated by van de Vooren and Botta [9]. If the medium is rotating with an angular velocity smaller than that of the disk, a double-deck occurs at the edge of the disk, while there will be a Stewartson layer at some distance from the edge. This will be the subject of a later publication.

2. General equations

For an axially symmetric configuration the dimensionless equations of motion are in an inertial system of reference

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + E\left\{\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right\},$$
$$u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = E\left\{\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right\}, \tag{2.1}$$
$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + E\left\{\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right\}.$$

u, v and w are the radial, azimuthal and axial velocities, resp. and p is the pressure. In order to satisfy the equation of continuity a streamfunction ψ is introduced by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{2.2}$$

With the scaled variables

$$\tilde{r} = (r-1)\mathrm{E}^{-1/2}\,, \ \tilde{z} = z\mathrm{E}^{-1/2} \tag{2.3}$$

and the substitutions

$$\psi = \frac{1}{2}\,\varepsilon \mathrm{E}^{1/2} h(\tilde{r}, \tilde{z})\,, \ u = \frac{1}{2}\,\varepsilon\,\frac{\partial h}{\partial \tilde{z}}\,(\tilde{r}, \tilde{z})\,, \ v = r - \varepsilon g(\tilde{r}, \tilde{z})\,,$$
$$w = -\frac{1}{2}\,\varepsilon\,\frac{\partial h}{\partial \tilde{r}}\,(\tilde{r}, \tilde{z})\,, \ p = \frac{1}{2}\,(1-2\varepsilon)r^2 + \varepsilon \mathrm{E}^{1/2}\tilde{p}(\tilde{r}, \tilde{z})\,, \tag{2.4}$$

we obtain, when linearizing in the small Rossby number ε, the following set of equations

$$\Delta\,\frac{\partial h}{\partial \tilde{z}} - 2\,\frac{\partial \tilde{p}}{\partial \tilde{r}} - 4g + 4 = 0\,,$$
$$\Delta g + \frac{\partial h}{\partial \tilde{z}} = 0\,, \tag{2.5}$$
$$\Delta\,\frac{\partial h}{\partial \tilde{r}} + 2\,\frac{\partial \tilde{p}}{\partial \tilde{z}} = 0\,,$$

where

$$\Delta = \frac{\partial^2}{\partial \tilde{r}^2} + \frac{\partial^2}{\partial \tilde{z}^2}\,.$$

In order that the linearization in ε is valid, we should have $\varepsilon < O(\mathrm{E}^{1/2})$. Eliminating $\tilde{p}$ from (2.5) and introducing the vorticity

$$\gamma = \frac{\partial w}{\partial r} - \frac{\partial u}{\partial z} \text{ or } \gamma = -\frac{1}{2}\,\varepsilon \mathrm{E}^{-1/2}\omega(\tilde{r}, \tilde{z})$$

with

$$\omega = \Delta h \tag{2.6}$$

the final set of equations becomes

$$\Delta h - \omega = 0\,,$$
$$\Delta g + \frac{\partial h}{\partial \tilde{z}} = 0\,, \tag{2.7}$$
$$\Delta\omega - 4\,\frac{\partial g}{\partial \tilde{z}} = 0\,.$$

3. Boundary conditions

$$\tilde{z} = 0\,, \ \tilde{r} < 0\colon\ h = 0,\ g = 0,\ \frac{\partial h}{\partial \tilde{z}} = 0\,,$$
$$\tilde{z} = 0\,, \ \tilde{r} > 0\colon\ h = 0\,,\ \frac{\partial g}{\partial \tilde{z}} = 0\,,\ \omega = 0\,. \tag{3.1}$$

The solution of the equations for the Ekman layer (see [1]) provides the boundary conditions at $\tilde{r} \to -\infty$

$$h \to 1 - \mathrm{e}^{-\tilde{z}}(\sin \tilde{z} + \cos \tilde{z}) ,$$
$$g \to 1 - \mathrm{e}^{-\tilde{z}} \cos \tilde{z} , \tag{3.2}$$
$$\omega \to -2\mathrm{e}^{-\tilde{z}}(\sin \tilde{z} - \cos \tilde{z}) .$$

For $\tilde{r} \to \infty$ we have undisturbed flow

$$h \to 0 , \; g \to 1 , \; \omega \to 0 . \tag{3.3}$$

Finally, for $\tilde{z} \to \infty$, the flow should match to the flow in the Stewartson layer for $z \to 0$, $r_1 \to 0$, where r_1 is the radial coordinate, $r_1 = (r-1)\mathrm{E}^{-1/3}$. According to [1] the solution for h, valid in the whole r_1, z-plane is

$$r_1 \geq 0: \; h(r_1, z) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin y}{y} \, \mathrm{e}^{-y^3\tau/2} \, \mathrm{d}y , \; \tau = \frac{z}{r_1^3} , \tag{3.4}$$
$$h(-r_1, z) = 1 - h(r_1, z) .$$

The solution for g, again valid in the whole Stewartson plane is

$$r_1 \geq 0: \; g(r_1, z) = 1 + \frac{\mathrm{E}^{1/6}}{2\pi r_1} \int_0^\infty \sin y \, \mathrm{e}^{-y^3\tau/2} \, \mathrm{d}y , \tag{3.5}$$
$$g(-r_1, z) = 2 - g(r_1, z) .$$

Writing the vorticity γ in the Stewartson variables we obtain

$$\gamma = \frac{\partial w}{\partial r} - \frac{\partial u}{\partial z} = \varepsilon \mathrm{E}^{-1/6} \frac{\partial w_1}{\partial r_1} ,$$

since $u = \varepsilon \mathrm{E}^{1/2} u_1$, $w = \varepsilon \mathrm{E}^{1/6} w_1$, $\partial/\partial r = \mathrm{E}^{-1/3} \partial/\partial r_1$. The term $\partial u/\partial z$ is $O(\mathrm{E}^{1/2})$ and, hence, can be neglected with regard to the term $\partial w/\partial r$. Taking the result for w_1 from [1] which is

$$r_1 \geq 0: \; w_1(r_1, z) = \frac{3z}{4\pi r_1^4} \int_0^\infty y^2 \sin y \, \mathrm{e}^{-y^3\tau/2} \, \mathrm{d}y ,$$
$$w_1(-r_1, z) = w_1(r_1, z) ,$$

we find

$$r_1 \geq 0: \; \gamma(r_1, z) = -\frac{\varepsilon \mathrm{E}^{-1/6}}{2\pi r_1^2} \int_0^\infty y \sin y \, \mathrm{e}^{-y^3\tau/2} \, \mathrm{d}y ,$$

$$\gamma(-r_1, z) = -\gamma(r_1, z) .$$

Hence

$$r_1 \geq 0:\ \omega(r_1, z) = \frac{E^{1/3}}{\pi r_1^2} \int_0^\infty y \sin y\, e^{-y^3\tau/2}\, dy\,, \tag{3.6}$$

$$\omega(-r_1, z) = -\omega(r_1, z)\,.$$

The expressions (3.4), (3.5) and (3.6) must be transformed from the r_1, z-coordinates to the $\tilde{r}$, $\tilde{z}$-coordinates. Since $z = \tilde{z}E^{1/2}$ and $r_1 = \tilde{r}E^{1/6}$, it is clear that

$$\tau = z/r_1^3 = \tilde{z}/\tilde{r}^3$$

and τ can be retained using the new variables. Hence, we have for $\tilde{z} \to \infty$

$$\begin{aligned} &\tilde{r} \geq 0:\ h(\tilde{r}, \tilde{z}) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin y}{y}\, e^{-y^3\tau/2}\, dy\,, \\ &g(\tilde{r}, \tilde{z}) = 1 + \frac{1}{2\pi\tilde{r}} \int_0^\infty \sin y\, e^{-y^3\tau/2}\, dy\,, \\ &\omega(\tilde{r}, \tilde{z}) = \frac{1}{\pi\tilde{r}^2} \int_0^\infty y \sin y\, e^{-y^3\tau/2}\, dy\,. \end{aligned} \tag{3.7}$$

Finally, the boundary conditions for $\tilde{z} \to \infty$ become

$$\begin{aligned} &\tilde{r} \geq 0:\ h(\tilde{r}, \infty) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin y}{y}\, e^{-y^3\tau/2}\, dy\,, \\ &h(-\tilde{r}, \infty) = 1 - h(\tilde{r}, \infty),\ g(\tilde{r}, \infty) = g(-\tilde{r}, \infty) = 1,\ \omega(\tilde{r}, \infty) = \omega(-\tilde{r}, \infty) = 0\,. \end{aligned} \tag{3.8}$$

4. The different regions in the $\tilde{r}$, $\tilde{z}$-plane

It follows from (3.8) that for finite $\tilde{r}$ the boundary condition of h at $\tilde{z} \to \infty$ gives the value 1/2. The transition from this value to the value 1 at $\tilde{r} \to -\infty$ and to 0 at $\tilde{r} \to \infty$ occurs for infinitely large values of $|\tilde{r}|$. Therefore, we have to consider τ instead of $\tilde{r}$ as variable for $\tilde{z} \to \infty$. However, since all lines $\tau = \text{constant}$ pass through the point $\tilde{r} = 0$, $\tilde{z} = 0$, we take τ only as variable for $\tilde{z}$ larger than some large value $\tilde{z}_m$. This occurs in region III (see Fig. 2). A second complication is that at the edge of the disk $\tilde{r} = 0$, $\tilde{z} = 0$, the vorticity becomes infinitely large. In order to investigate the behaviour of the functions h, g and ω near this point, we introduce both cylindrical and parabolic coordinates by

$$\tilde{r} + i\tilde{z} = \rho\, e^{i\theta} = \left(\frac{\xi + i\eta}{p}\right)^2,\ p \text{ integer} \tag{4.1}$$

Since for $\rho \to 0$, $h = O(\rho^{3/2})$, $g = O(\rho^{1/2})$ and $\omega = O(\rho^{-1/2})$ the equations simplify to

$$\Delta h = \omega\,,\ \Delta g = 0\,,\ \Delta\omega = 0\,, \tag{4.2}$$

since the terms containing first derivatives to $\tilde{z}$ are of less importance for $\rho \to 0$ than the other terms.

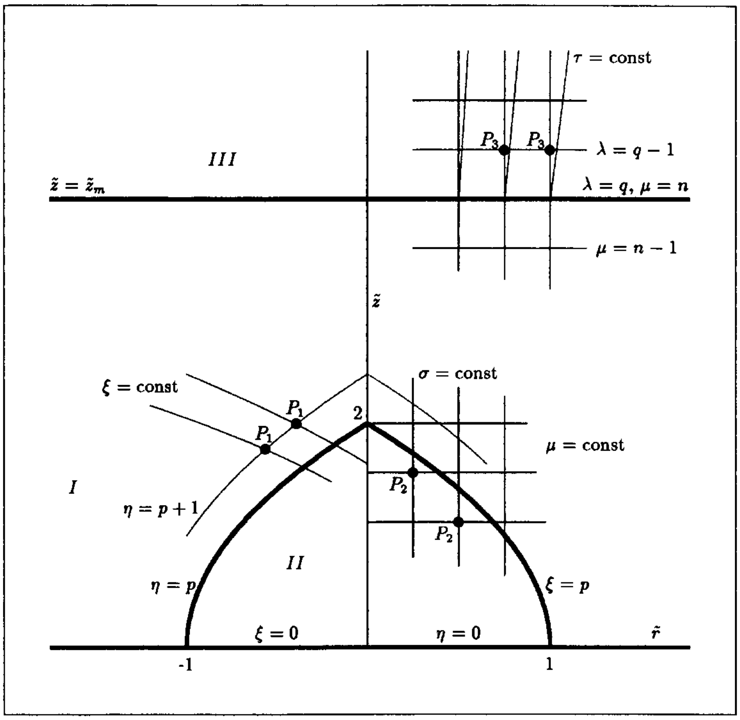

Fig. 2. The different regions in the $\tilde{r}$, $\tilde{z}$-plane.

The solution of equations (4.2) with the boundary conditions (3.1) for $\tilde{z}=0$ is $(\rho \to 0)$:

$$
\begin{aligned}
h &= C_1\rho^{3/2}\sin\theta\cos\theta/2 = 2C_1\xi^2\eta/p^3 , \\
g &= C_2\rho^{1/2}\cos\theta/2 = C_2\xi/p , \\
\omega &= C_1\rho^{-1/2}\sin\theta/2 = C_1\frac{\eta}{\xi^2+\eta^2}\,p
\end{aligned}
\tag{4.3}
$$

The pressure $\tilde{p}$ follows from the first and third equations (2.5) which can be written as

$$\frac{\partial\omega}{\partial\tilde{z}} - 2\frac{\partial\tilde{p}}{\partial\tilde{r}} - 4g + 4 = 0 , \quad \frac{\partial\omega}{\partial\tilde{r}} + 2\frac{\partial\tilde{p}}{\partial\tilde{z}} = 0 .$$

Near $\rho = 0$ the terms containing ω are $O(\rho^{-3/2})$ and hence, $-4g+4$ can be neglected. The remaining equations are of Cauchy-Riemann type which means that $\omega + 2i\tilde{p}$ should be a function of $\tilde{r} + i\tilde{z}$. With the known value of ω we find

$$2\tilde{p} = -C_1\rho^{-1/2}\cos\theta/2 .$$

More complete expressions for ω and $\tilde{p}$ are

$$
\begin{aligned}
\omega &= C_1\rho^{-1/2}\sin\theta/2 + C_3\rho^{1/2}\sin\theta/2 + O(\rho^{3/2}) , \\
2\tilde{p} &= -C_1\rho^{-1/2}\cos\theta/2 - C_3\rho^{1/2}\cos\theta/2 + C_4 + 4\rho\cos\theta + O(\rho^{3/2}) .
\end{aligned}
$$

Both the vorticity and the pressure have a singularity at the origin. Since C_1 is positive, the pressure is negative near the origin except possibly at the disk.

Within the region II, defined by $0 \leq \xi \leq p$, $0 \leq \eta \leq p$ the parabolic coordinates ξ, η will be taken as independent variables. The dependent variable ω becoming infinitely large at $\rho \to 0$ will be replaced by $k = \rho\omega$ in region II. Thus

$$k = C_1 \rho^{1/2} \sin \theta/2 = C_1 \eta/p . \tag{4.4}$$

The remaining part of the $\tilde{r}$, $\tilde{z}$-plane will be denoted as region I.

5. The equations in the various regions

Region I. The variable $\tilde{r}$ will be replaced by a finite variable σ in which we will take a constant mesh size. The transformation should also be such that the values of $\tau = \tilde{z}_m/\tilde{r}^3$ lead to a function $h(\tilde{r}, \infty)$, given by (3.8), which changes gradually with σ. This is realized by the transformation

$$\tilde{r} = \frac{\sigma\sqrt{A + \sigma^2}}{m^2 - \sigma^2} , \quad -m \leq \sigma \leq m . \tag{5.1}$$

We want that $\tilde{r} = 1$ corresponds to $\sigma = 0.3\, m$ which leads to $A = 82\, m^2/9$ (m mod $10 = 0$). In reverse, σ follows from $\tilde{r}$ by aid of the quadratic equation

$$\sigma^4 + a_1\sigma^2 + a_2 = 0 , \text{ where } a_1 = \frac{A + 2m^2\tilde{r}^2}{1 - \tilde{r}^2} , \; a_2 = -\frac{m^4\tilde{r}^2}{1 - \tilde{r}^2} .$$

In the solution

$$\sigma^2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \tag{5.2}$$

we should take the plus sign for $\tilde{r}^2 > 1$ and the minus sign for $\tilde{r}^2 < 1$.

The variable $\tilde{z}$ is replaced by a variable μ. This transformation is taken in such a way that for equal steps in μ, we obtain more points for small $\tilde{z}$ (in the boundary layer). We take

$$\tilde{z} = \tilde{z}_m \frac{\sinh \beta\mu/n}{\sinh \beta} , \quad 0 \leq \mu \leq n . \tag{5.3}$$

We want that $\tilde{z} = 2$ corresponds to the integer value of $\mu = n/2$ (n even). This determines the value of $\tilde{z}_m$. For $\beta = 5$ we find $\tilde{z}_m = 24.5291579$, which is well outside the Ekman boundary layer. The inverse formula is

$$\mu = \frac{n}{\beta} \ln\left\{ \frac{\tilde{z}}{\tilde{z}_m} \sinh \beta + \sqrt{\left(\frac{\tilde{z}}{\tilde{z}_m} \sinh \beta\right)^2 + 1} \right\} . \tag{5.4}$$

In region I equations (2.7) written in the σ, μ-variables become

$$c_1(\sigma)\frac{\partial^2 h}{\partial\sigma^2}+c_2(\sigma)\frac{\partial h}{\partial\sigma}+c_3(\mu)\frac{\partial^2 h}{\partial\mu^2}+c_4(\mu)\frac{\partial h}{\partial\mu}-\omega=0\,,$$

$$c_1(\sigma)\frac{\partial^2 g}{\partial\sigma^2}+c_2(\sigma)\frac{\partial g}{\partial\sigma}+c_3(\mu)\frac{\partial^2 g}{\partial\mu^2}+c_4(\mu)\frac{\partial g}{\partial\mu}+c_5(\mu)\frac{\partial h}{\partial\mu}=0\,, \tag{5.5}$$

$$c_1(\sigma)\frac{\partial^2 \omega}{\partial\sigma^2}+c_2(\sigma)\frac{\partial \omega}{\partial\sigma}+c_3(\mu)\frac{\partial^2 \omega}{\partial\mu^2}+c_4(\mu)\frac{\partial \omega}{\partial\mu}-4c_5(\mu)\frac{\partial g}{\partial\mu}=0\,,$$

where

$$c_1(\sigma)=\frac{1}{(\mathrm{d}\tilde{r}/\mathrm{d}\sigma)^2}\,,\quad c_2(\sigma)=-\frac{\mathrm{d}^2\tilde{r}/\mathrm{d}\sigma^2}{(\mathrm{d}\tilde{r}/\mathrm{d}\sigma)^3}\,,$$

$$c_3(\mu)=\frac{1}{(\mathrm{d}\tilde{z}/\mathrm{d}\mu)^2}\,,\quad c_4(\mu)=-\frac{\mathrm{d}^2\tilde{z}/\mathrm{d}\mu^2}{(\mathrm{d}\tilde{z}/\mathrm{d}\mu)^3}\,,\quad c_5(\mu)=\frac{1}{\mathrm{d}\tilde{z}/\mathrm{d}\mu}\,. \tag{5.6}$$

Region II. By the introduction of p we have obtained the following correspondence of points

$$\begin{array}{llllll}
\tilde{r}=-1\,, & \tilde{z}=0\,, & \sigma=-0.3\,m\,, & \mu=0\,, & \xi=0\,, & \eta=p\,.\\
\tilde{r}=0\,, & \tilde{z}=2\,, & \sigma=0\,, & \mu=0.5\,n\,, & \xi=p\,, & \eta=p\,.\\
\tilde{r}=1\,, & \tilde{z}=0\,, & \sigma=0.3\,m\,, & \mu=0\,, & \xi=p\,, & \eta=0\,.
\end{array}$$

These corner points of region II fit into the mesh points of region I. In the ξ, η-variables and introducing k according to (4.4), equations (2.7) become in region II

$$\frac{\partial^2 h}{\partial\xi^2}+\frac{\partial^2 h}{\partial\eta^2}-\frac{4}{p^2}\,k=0\,,$$

$$\frac{\partial^2 g}{\partial\xi^2}+\frac{\partial^2 g}{\partial\eta^2}+\frac{2}{p^2}\left(\eta\frac{\partial h}{\partial\xi}+\xi\frac{\partial h}{\partial\eta}\right)=0\,, \tag{5.7}$$

$$\frac{\partial^2 k}{\partial\xi^2}+\frac{\partial^2 k}{\partial\eta^2}-\frac{4}{\xi^2+\eta^2}\left(\xi\frac{\partial k}{\partial\xi}+\eta\frac{\partial k}{\partial\eta}-k\right)-\frac{8(\xi^2+\eta^2)}{p^4}\left(\eta\frac{\partial g}{\partial\xi}+\xi\frac{\partial g}{\partial\eta}\right)=0\,.$$

Region III. With a slight deviation from what has been mentioned in Sec. 4, we will take as variables in region III $\tau_1=\tau^{1/3}=\tilde{z}^{1/3}/\tilde{r}$ and $\tilde{z}$. In these variables equations (2.7) become

$$\Delta_{\tau_1,\tilde{z}}h-\omega=0\,,$$

$$\Delta_{\tau_1,\tilde{z}}g+\frac{\tau_1}{3\tilde{z}}\frac{\partial h}{\partial\tau_1}+\frac{\partial h}{\partial\tilde{z}}=0\,, \tag{5.8}$$

$$\Delta_{\tau_1,\tilde{z}}\omega-\frac{4\tau_1}{3\tilde{z}}\frac{\partial g}{\partial\tau_1}-4\frac{\partial g}{\partial\tilde{z}}=0\,,$$

where

$$\Delta_{\tau_1,\tilde{z}}=\left(\frac{\tau_1^4}{\tilde{z}^{2/3}}+\frac{\tau_1^2}{9\tilde{z}^2}\right)\frac{\partial^2}{\partial r_1^2}+\frac{2\tau_1}{3\tilde{z}}\frac{\partial^2}{\partial\tau_1\partial\tilde{z}}+\frac{\partial^2}{\partial\tilde{z}^2}+\left(\frac{2\tau_1^3}{\tilde{z}^{2/3}}-\frac{2\tau_1}{9\tilde{z}^2}\right)\frac{\partial}{\partial\tau_1}\,. \tag{5.9}$$

However, we again want variables in which we can take a constant mesh size. Let the point of intersection of the curves τ_1 = constant and $\tilde{z} = \tilde{z}_m$ be denoted by $\tilde{r} = \tilde{r}_m$ (see Fig. 2). This point lies at the boundary between regions I and III and therefore can, in agreement with (5.1), also be denoted by a σ-value. This means that we can replace the τ_1-variable by the σ-variable according to

$$\tau_1 = \frac{\tilde{z}^{1/3}}{\tilde{r}} = \frac{\tilde{z}_m^{1/3}}{\tilde{r}_m} = \frac{\tilde{z}_m^{1/3}(m^2 - \sigma^2)}{\sigma\sqrt{A + \sigma^2}} . \tag{5.10}$$

Then

$$\frac{\partial}{\partial \tau_1} = \frac{1}{\mathrm{d}\tau_1/\mathrm{d}\sigma} \frac{\partial}{\partial \sigma} \text{ and } \frac{\partial^2}{\partial \tau_1^2} = \frac{1}{(\mathrm{d}\tau_1/\mathrm{d}\sigma)^2} \frac{\partial^2}{\partial \sigma^2} - \frac{\mathrm{d}^2\tau_1/\mathrm{d}\sigma^2}{(\mathrm{d}\tau_1/\mathrm{d}\sigma)^3} \frac{\partial}{\partial \sigma} .$$

The variable $\tilde{z}$ will be replaced by a variable λ defined by

$$\lambda = \frac{B}{\tilde{z}^{1/3}} . \tag{5.11}$$

It follows from (3.7) that $g = 1 + O(\tilde{z}^{-1/3})$ and $\omega = O(\tilde{z}^{-2/3})$ for $\tilde{z} \to \infty$ and τ constant. Hence, by accepting (5.11), we shall have $g = 1 + O(\lambda)$, $\omega = O(\lambda^2)$ for $\lambda \to 0$. We take $\lambda = q$ when $\tilde{z} = \tilde{z}_m$ with q integer. Hence

$$q = \frac{B}{\tilde{z}_m^{1/3}} ,$$

which determines B when the integer q has been chosen. λ varies from 0 to q in region III. With

$$\frac{\partial}{\partial \tilde{z}} = -\frac{\lambda}{3\tilde{z}} \frac{\partial}{\partial \lambda} \text{ and } \frac{\partial^2}{\partial \tilde{z}^2} = \frac{\lambda^2}{9\tilde{z}^2} \frac{\partial^2}{\partial \lambda^2} + \frac{\lambda}{9\tilde{z}^2} \frac{\partial}{\partial \lambda} ,$$

eqs (5.8) finally become

$$\begin{aligned}
&c_6(\sigma, \lambda) \frac{\partial^2 h}{\partial \sigma^2} + c_7(\sigma, \lambda) \frac{\partial^2 h}{\partial \sigma \partial \lambda} + \lambda c_8(\lambda) \frac{\partial^2 h}{\partial \lambda^2} + c_9(\sigma, \lambda) \frac{\partial h}{\partial \sigma} + 4c_8(\lambda) \frac{\partial h}{\partial \lambda} - \omega = 0 , \\
&c_6(\sigma, \lambda) \frac{\partial^2 g}{\partial \sigma^2} + c_7(\sigma, \lambda) \frac{\partial^2 g}{\partial \sigma \partial \lambda} + \lambda c_8(\lambda) \frac{\partial^2 g}{\partial \lambda^2} + c_9(\sigma, \lambda) \frac{\partial g}{\partial \sigma} + 4c_8(\lambda) \frac{\partial g}{\partial \lambda} + c_{10}(\sigma, \lambda) \frac{\partial h}{\partial \sigma} \\
&\qquad + c_{11}(\lambda) \frac{\partial h}{\partial \lambda} = 0 , \\
&c_6(\sigma, \lambda) \frac{\partial^2 \omega}{\partial \sigma^2} + c_7(\sigma, \lambda) \frac{\partial^2 \omega}{\partial \sigma \partial \lambda} + \lambda c_8(\lambda) \frac{\partial^2 \omega}{\partial \lambda^2} + c_9(\sigma, \lambda) \frac{\partial \omega}{\partial \sigma} + 4c_8(\lambda) \frac{\partial \omega}{\partial \lambda} \\
&\qquad - 4c_{10}(\sigma, \lambda) \frac{\partial g}{\partial \sigma} - 4c_{11}(\lambda) \frac{\partial g}{\partial \lambda} = 0 ,
\end{aligned} \tag{5.12}$$

where

$$c_6(\sigma, \lambda) = \frac{\tau_1^2\left(\frac{\lambda^2\tau_1^2}{B^2} + \frac{1}{9\tilde{z}^2}\right)}{(\mathrm{d}\tau_1/\mathrm{d}\sigma)^2}, \quad c_7(\sigma, \lambda) = -\frac{2\lambda\tau_1}{9\tilde{z}^2\,\mathrm{d}\tau_1/\mathrm{d}\sigma},$$

$$c_8(\lambda) = \frac{\lambda}{9\tilde{z}^2}, \quad c_9(\sigma, \lambda) = \left\{2\tau_1\left(\frac{\lambda^2\tau_1^2}{B^2} - \frac{1}{9\tilde{z}^2}\right) - c_6(\sigma, \lambda)\frac{\mathrm{d}^2\tau_1/\mathrm{d}\sigma^2}{(\mathrm{d}\tau_1/\mathrm{d}\sigma)^2}\right\}\frac{1}{\mathrm{d}\tau_1/\mathrm{d}\sigma}, \tag{5.13}$$

$$c_{10}(\sigma, \lambda) = \frac{\tau_1}{3\tilde{z}\,\mathrm{d}\tau_1/\mathrm{d}\sigma}, \quad c_{11}(\lambda) = -\frac{\lambda}{3\tilde{z}}.$$

6. Asymptotic behaviour for $\tilde{z} \to 0$

Elimination of the functions g and ω from (2.7) leads to

$$\Delta\Delta\Delta h + 4\frac{\partial^2 h}{\partial \tilde{z}^2} = 0,$$

or

$$\frac{\partial^6 h}{\partial \tilde{r}^6} + 3\frac{\partial^6 h}{\partial \tilde{r}^4 \partial \tilde{z}^2} + 3\frac{\partial^6 h}{\partial \tilde{r}^2 \partial \tilde{z}^4} + \frac{\partial^6 h}{d\tilde{z}^6} + 4\frac{\partial^2 h}{\partial \tilde{z}^2} = 0. \tag{6.1}$$

For $\tilde{z} \to \infty$ we have according to (3.8)

$$h(\tilde{r}, \tilde{z}) = f(\tau) \text{ with } \tau = \tilde{z}/\tilde{r}^3.$$

We investigate the asymptotics for $\tilde{z} \to \infty$ and τ constant. Then

$$\frac{\partial h}{\partial \tilde{z}} = \frac{\tau f'(\tau)}{\tilde{z}} \text{ and } \frac{\partial h}{\partial \tilde{r}} = -\frac{3\tau^{4/3} f'(\tau)}{\tilde{z}^{1/3}}.$$

Each differentiation to $\tilde{z}$ produces a factor $O(\tilde{z}^{-1})$ and each differentiation to $\tilde{r}$ a factor $O(\tilde{z}^{-1/3})$. Hence, the most important terms in (6.1) for $\tilde{z} \to \infty$ are

$$\frac{\partial^6 h}{\partial \tilde{r}^6} + 4\frac{\partial^2 h}{\partial \tilde{z}^2} = 0 \tag{6.2}$$

and this is exactly the equation which yields (3.8) as a solution, see [1]. However, the general solution of (6.2) may be written as

$$h(\tilde{r}, \tilde{z}) = \int_{-\infty}^{\infty} A(\omega)\,\mathrm{e}^{-i\omega\tilde{r}}\,\mathrm{e}^{-|\omega|^3\tilde{z}/2}\,\mathrm{d}\omega. \tag{6.3}$$

Separating $A(\omega)$ in an even part $A_e(\omega)$ and an odd part $A_o(\omega)$, we obtain

$$h(\tilde{r}, \tilde{z}) = \int_0^{\infty} A_e(\omega)\cos\omega\tilde{r}\,\mathrm{e}^{-\omega^3\tilde{z}/2}\,\mathrm{d}\omega + \int_0^{\infty} A_o(\omega)\sin\omega\tilde{r}\,\mathrm{e}^{-\omega^3\tilde{z}/2}\,\mathrm{d}\omega.$$

Since this solution $h(\tilde{r}, \tilde{z})$ makes only sense for $\tilde{z} \to \infty$, it is clear that the values of the integrals are determined by small values of ω. The expressions of $A_e(\omega)$ and $A_o(\omega)$ for $\omega \to 0$ contain general terms $c_{2n}\omega^{2n}$ and $c_{2n+1}\omega^{2n+1}$, respectively. The general term of the first integral is

$$c_{2n}\int_0^\infty \omega^{2n}\cos\omega\tilde{r}\,\mathrm{e}^{-\omega^3\tilde{z}/2}\,\mathrm{d}\omega$$

or, by taking $\omega\tilde{r}=y$ and $1/\tilde{r}=(\tau/\tilde{z})^{1/3}$, $\tilde{r}$ and $\tau>0$

$$h_{2n}(\tau,\tilde{z})=c_{2n}\left(\frac{\tau}{\tilde{z}}\right)^{\frac{2n+1}{3}}\int_0^\infty y^{2n}\cos y\,\mathrm{e}^{-y^3\tau/2}\,\mathrm{d}y\,. \tag{6.4}$$

In the same way the second integral leads to

$$h_{2n+1}(\tau,\tilde{z})=c_{2n+1}\left(\frac{\tau}{\tilde{z}}\right)^{\frac{2n+2}{3}}\int_0^\infty y^{2n+1}\sin y\,\mathrm{e}^{-y^3\tau/2}\,\mathrm{d}y\,. \tag{6.5}$$

For negative values of τ we have

$$h_{2n}(\tau,\tilde{z})=h_{2n}(-\tau,\tilde{z}),\ h_{2n+1}(\tau,\tilde{z})=-h_{2n+1}(-\tau,\tilde{z})\,.$$

It follows that the asymptotic expansion of $h(\tau,\tilde{z})$ for $\tilde{z}\to\infty$ will be a series of which the consecutive terms diminish by a factor $\tilde{z}^{-1/3}$ provided the coefficients c_{2n} are unequal to 0.

However, solutions in the $\tilde{r}$, $\tilde{z}$-region connecting the Ekman and the Stewartson layers must be matched for $\tilde{z}\to\infty$ to solutions in the Stewartson layer for $z\to 0$. Since τ remains unaltered and $\tilde{z}=\mathrm{E}^{-1/2}z$, we have in the Stewartson layer for $z\to 0$

$$h_{2n}(\tau,z)=c_{2n}\mathrm{E}^{\frac{2n+1}{3}}\left(\frac{\tau}{z}\right)^{\frac{2n+1}{3}}\int_0^\infty y^{2n}\cos y\,\mathrm{e}^{-y^3\tau/2}\,\mathrm{d}y\,,$$

$$h_{2n+1}(\tau,z)=c_{2n+1}\mathrm{E}^{\frac{2n+2}{6}}\left(\frac{\tau}{z}\right)^{\frac{2n+2}{3}}\int_0^\infty y^{2n+1}\sin y\,\mathrm{e}^{-y^3\tau/2}\,\mathrm{d}y\,.$$

This agrees with

$$h_{2n}(r_1,z)=c_{2n}\mathrm{E}^{\frac{2n+1}{6}}\int_0^\infty \omega^{2n}\cos\omega r_1\,\mathrm{e}^{-\omega^3z/2}\,\mathrm{d}\omega\,, \tag{6.6}$$

$$h_{2n+1}(r_1,z)=c_{2n+1}\mathrm{E}^{\frac{2n+2}{6}}\int_0^\infty \omega^{2n+1}\sin\omega r_1\,\mathrm{e}^{-\omega^3z/2}\,\mathrm{d}\omega\,. \tag{6.7}$$

For $z\to 0$ and $r_1\neq 0$ we find

$$h_{2n}(r_1,0)=0\,,\ h_{2n+1}(r_1,0)=0\,.$$

For the original problem of the Stewartson layer, see [1], we had for $h(r_1,z)$ prescribed values at the boundary $z=0$, $r_1\neq 0$. It appears that the additional solutions (6.6) and (6.7) do not disturb these prescribed boundary values. They are homogeneous solutions of $O(\mathrm{E}^{\kappa/6})$ with $\kappa=1,2,\ldots$, which have to be added to the solution given in [1]. Thus we have in the Stewartson layer

$$\psi=\varepsilon\mathrm{E}^{1/2}h=\varepsilon(\mathrm{E}^{1/2}\psi_1+\mathrm{E}^{2/3}\psi_{3/2}+\mathrm{E}^{5/6}\psi_2+\cdots)\,. \tag{6.8}$$

It follows that ψ_1 is not affected by the homogeneous solutions, that $\psi_{3/2}$ is exclusively determined by the homogeneous solution (6.6) with $n = 0$ and that ψ_2 will contain, besides the contribution found already in [1], also a term (6.7) with $n = 0$.

The results from the numerical calculations show that for $\tilde{z} \to \infty$ the deviation of h from its value at $\tilde{z} = \infty$ is indeed $O(\tilde{z}^{-1/3})$. There is no reason to assume that any of the coefficients c_κ vanishes. Hence, (6.8) is the correct expansion for $E \to 0$ in the Stewartson layer.

It might be added that if in (6.1) terms of $O(\tilde{z}^{-10/3})$ are retained, we have

$$\frac{\partial^6 h_2}{\partial \tilde{r}^6} + 4 \frac{\partial^2 h_2}{\partial \tilde{z}^2} = -3 \frac{\partial^6 h}{\partial \tilde{r}^4 \partial \tilde{z}^2} ,$$

which gives a solution $h_2 = O(\tilde{z}^{-4/3})$ for $\tilde{z} \to \infty$.

7. The numerical method

The differential equations (5.5), (5.7) and (5.12) in the various regions are approximated by difference equations using the following meshes:

region I	$\Delta\sigma = 1$,	$\Delta\mu = 1$,	$-m \leq \sigma \leq m$,	$0 \leq \mu \leq n$,
region II	$\Delta\xi = 1$,	$\Delta\eta = 1$,	$0 \leq \xi \leq p$,	$0 \leq \eta \leq p$.
region III	$\Delta\sigma = 1$,	$\Delta\lambda = 1$,	$-m \leq \sigma \leq m$,	$0 \leq \lambda \leq q$.

Calculations were performed for

case 1. $m = 10$, $n = 40$, $p = 10$, $q = 10$.
case 2. $m = 20$, $n = 80$, $p = 20$, $q = 20$.

Line iteration along lines of constant σ was used in regions I and III with simultaneous calculation of the three dependent variables h, g and ω. The lines were swept in the direction of increasing σ.

Since the mesh geometries in regions I and III are different, we had to take special precaution when applying the difference equations at boundary points $(\tilde{r}, \tilde{z}_m)$. We used in these points the (extrapolated) mesh of region I and in this way had to obtain the variables h, g and ω in points P_3 of region III with the same value of $\tilde{r}$ by linear interpolation along the line $\lambda = q - 1$.

Since the boundary between regions I and II does not fit into the mesh of region I, values of h, g and ω in points P_2 in region II and belonging to the mesh of region I had to be obtained by linear interpolation in a (ξ, η)-rectangle. On behalf of the calculation in region II also h, g and ω in points P_1, fitting into the mesh of region II had to be found by linear interpolation in a (σ, μ)-rectangle. In region II the line iteration was performed along lines of constant η, sweeping from $\eta = p$ to $\eta = 0$. This iteration was executed each time after having arrived with the iteration in regions I and III at $\sigma = m - 1$.

Two difficulties were encountered. The first one was that in region III for small values of λ (large values of $\tilde{z}$) the values h, g and ω showed an oscillation in such a way that results obtained at even values of λ did not fit with the results from odd λ-values. This was due to the fact that at large values of $\tilde{z}$ the first derivatives to $\tilde{z}$ become much larger than the second derivatives to $\tilde{z}$ and this destroys the coupling between the difference equations at even and odd values of λ.

To remedy this difficulty a staggered grid in regions I and III was introduced, taking the variables h and ω at even values of μ and λ and the variable g at odd values of μ and λ. Although this led to complications at the boundaries between regions I and III it removed the oscillation.

The second complication was that the function h showed a large derivative $\partial h/\partial\lambda$ for $\lambda \to 0$ ($\tilde{z} \to \infty$). In order to explain this, we consider the second equation (5.8). For $\tilde{z} \to \infty$ we have

$$h = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin y}{y}\,\mathrm{e}^{-y^3\tau_1^3/2}\,\mathrm{d}y\,,$$

$$g = 1 + \frac{\tau_1}{2\pi\tilde{z}^{1/3}}\int_0^\infty \sin y\,\mathrm{e}^{-y^3\tau_1^3/2}\,\mathrm{d}y\,.$$

The dominant terms in the second equation (5.8) for $\tilde{z} \to \infty$ are $O(\tilde{z}^{-1})$ and are produced by

$$\frac{\tau_1^4}{\tilde{z}^{2/3}}\frac{\partial^2 g}{\partial\tau_1^2} + \frac{2\tau_1^3}{\tilde{z}^{2/3}}\frac{\partial g}{\partial\tau_1} + \frac{\tau_1}{3\tilde{z}}\frac{\partial h}{\partial\tau_1}\,.$$

Substituting the formulae for h and g, given above, this expression is exactly 0. However, when approximating the derivatives by differences there will remain in $\partial h/\partial\tilde{z}$ a small term $O(\tilde{z}^{-1})$. Since

$$\frac{\partial h}{\partial\lambda} = -\frac{3\tilde{z}}{\lambda}\frac{\partial h}{\partial\tilde{z}}\,,\ \text{we find}\ \frac{\partial h}{\partial\lambda} = O(\lambda^{-1})$$

and this small erroneous term is responsible for the unsatisfactory behaviour found in the numerical calculation for $\lambda \to 0$.

In order to circumvent this difficulty we substitute in (5.12)

$$h = h_0 + h_1,\ g = g_0 + g_1,\ \omega = \omega_0 + \omega_1$$

with

$$h_0(\tau_1) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin y}{y}\,\mathrm{e}^{-y^3\tau_1^3/2}\,\mathrm{d}y\,,\ \tau_1 > 0\,,$$

$$g_0(\tau_1, \tilde{z}) = 1 + \frac{\tau_1}{2\pi\tilde{z}^{1/3}}\int_0^\infty \sin y\,\mathrm{e}^{-y^3\tau_1^3/2}\,\mathrm{d}y\,,\ \tau_1 > 0\,, \tag{7.1}$$

$$\omega_0(\tau_1, \tilde{z}) = \frac{\tau_1^2}{\pi\tilde{z}^{2/3}}\int_0^\infty y\sin y\,\mathrm{e}^{-y^3\tau_1^3/2}\,\mathrm{d}y\,,\ \tau_1 > 0\,.$$

and

$$h_0(-\tau_1) = 1 - h_0(\tau_1)\,,$$

$$g_0(-\tau_1, \tilde{z}) = 2 - g_0(\tau_1, \tilde{z})\,,$$

$$\omega_0(-\tau_1, \tilde{z}) = -\omega_0(\tau_1, \tilde{z})\,.$$

Equations (5.12) then become

$$c_6(\sigma,\lambda)\frac{\partial^2 h_1}{\partial\sigma^2}+c_7(\sigma,\lambda)\frac{\partial^2 h_1}{\partial\sigma\partial\lambda}+\lambda c_8(\lambda)\frac{\partial^2 h_1}{\partial\lambda^2}+c_9(\sigma,\lambda)\frac{\partial h_1}{\partial\sigma}+4c_8(\lambda)\frac{\partial h_1}{\partial\lambda}-\omega_1$$

$$=\frac{1}{9\pi\tilde{z}^2}\int_0^\infty(4\cos y-y\sin y)\,\mathrm{e}^{-y^3\tau_1^3/2}\mathrm{d}y\,,$$

$$c_6(\sigma,\lambda)\frac{\partial^2 g_1}{\partial\sigma^2}+c_7(\sigma,\lambda)\frac{\partial^2 g_1}{\partial\sigma\partial\lambda}+\lambda c_8(\lambda)\frac{\partial^2 g_1}{\partial\lambda^2}+c_9(\sigma,\lambda)\frac{\partial g_1}{\partial\sigma}+4c_8(\lambda)\frac{\partial g_1}{\partial\lambda}+c_{10}(\sigma,\lambda)\frac{\partial h_1}{\partial\sigma}$$

$$+c_{11}(\lambda)\frac{\partial h_1}{\partial\lambda}=-\frac{\tau_1}{9\pi\tilde{z}^{7/3}}\int_0^\infty(2\sin y+3y\cos y-\tfrac{1}{2}y^2\sin y)\,\mathrm{e}^{-y^3\tau_1^3/2}\mathrm{d}y\,,$$

$$c_6(\sigma,\lambda)\frac{\partial^2 \omega_1}{\partial\sigma^2}+c_7(\sigma,\lambda)\frac{\partial^2 \omega_1}{\partial\sigma\partial\lambda}+\lambda c_8(\lambda)\frac{\partial^2 \omega_1}{\partial\lambda^2}+c_9(\sigma,\lambda)\frac{\partial \omega_1}{\partial\sigma}+4c_8(\lambda)\frac{\partial \omega_1}{\partial\lambda}$$

$$-4c_{10}(\sigma,\lambda)\frac{\partial g_1}{\partial\sigma}-4c_{11}(\lambda)\frac{\partial g_1}{\partial\lambda}=-\frac{\tau_1^2}{9\pi\tilde{z}^{8/3}}\int_0^\infty(10y\sin y+8y^2\cos y-y^3\sin y)\,\mathrm{e}^{-y^3\tau_1^3/2}\mathrm{d}y\,.$$

These equations are replaced by difference equations, while the integrals in the right hand side have been calculated by Romberg integration.

8. Results

In both cases 1 and 2 the iteration has been continued until for all variables h, g and ω the differences between 2 consecutive iterationsteps were smaller than 10^{-5} for all points in the 3 regions. Although the convergence was rather slow, it may safely be assumed that the error due to the finite number of iterationsteps is smaller than 10^{-3}. The number of iterationsteps was 1000 for case 1 and 8000 for case 2. In the latter case a relaxation factor 0.7 had to be added in order to ensure convergence. The difference in the results of the two cases was of the order of 10^{-3}. The calculations were performed on a HP 9000/720 and took 32 minutes for case 2.

It was found that for $\tilde{z}\to\infty$ the difference

$$h(\tau,\tilde{z})-h_0(\tau) \tag{8.1}$$

behaved like $O(\tilde{z}^{-1/3})=O(\lambda)$. This implies that a homogeneous solution (6.4) with $n=0$ is present. When calculating the coefficient c_0 in (6.4) a number of digits cancel in the difference (8.1) but a not too accurate value $c_0=-0.17$ was obtained. However, no reliable value for c_1 could be calculated.

The occurrence of the additional solutions (6.4) and (6.5) means that the terms in the expansions for small E in the Stewartson layer increase like $\mathrm{E}^{1/6}$ as denoted for ψ by (6.8). This is an improvement compared with the series given in [1], where the terms in the expansions increase like $\mathrm{E}^{1/3}$. It also means that the term ψ_2 in [1] is incomplete and that the decrease of the axial velocity through the Stewartson layer is only qualitatively but not quantitatively correct.

The torque exerted by the rotating medium on the disk is determined by the tangential shear stress which is proportional to

$$\mathrm{E}^{1/2}\,\frac{\partial v}{\partial \tilde z} = -\varepsilon \mathrm{E}^{1/2}\,\frac{\partial g}{\partial \tilde z} \text{ for } \tilde z \to 0\,.$$

The value $\partial g/\partial \tilde z = 1$ in the Ekman layer leads to the torque, see [1]

$$M = -\frac{1}{2}\,\varepsilon \pi \rho \Omega^2 a^5 \mathrm{E}^{1/2}\,,$$

where a is the radius of the disk and ρ the density of the medium. In the region connecting the Ekman and Stewartson layers we have a different value of $\partial g/\partial \tilde z$ and since this region has a length $O(\mathrm{E}^{1/2})$ in radial direction, the deviation in $\partial g/\partial \tilde z$ from 1 gives rise to a torque $O(\mathrm{E})$.

The numerical calculations show that for $\tilde r \to -\infty$ we have

$$\tilde z = 0\colon \frac{\partial g}{\partial \tilde z} = 1 + \frac{c}{\tilde r}\,,$$

which is not integrable up to $\tilde r = -\infty$. The value of c could be estimated as being 0.18. The value of $\partial g/\partial \tilde z$ should be matched to the value in the Ekman layer, which then should be

$$\tilde z = 0\colon \frac{\partial g}{\partial \tilde z} = 1 - \frac{c\mathrm{E}^{1/2}}{1 - r}\,. \tag{8.2}$$

The second term must follow from second order boundary layer theory, which also will provide a term $O(\mathrm{E})$ in the torque. It is interesting to see that formula (6.1) from [1] gives for the tangential flow velocity g outside the Ekman layer and for $r \uparrow 1$ the expression

$$\tilde z = \infty\colon g = 1 - \frac{\mathrm{E}^{1/2}}{2\pi(1 - r)}\,. \tag{8.3}$$

This gives full confidence that a second order boundary layer calculation with the outer flow (8.3) will lead to a shear stress in accordance with (8.2).

Figure 3 shows streamlines h = constant in the $\tilde r$, $\tilde z$-plane. The negative pressure near the origin gives there an acceleration of the flow, which is too small to be visible in the figure. For positive $\tilde r$ the pressure increases and this causes the radially incoming flow to be turned in axial direction before it reaches $\tilde r \sim 4$. For larger values of $\tilde r$ there is a weak recirculating flow.

For large values of $\tilde z$, h becomes constant along lines $\tau = \tilde z/\tilde r^3$ = constant. This implies that $h = 0.5$ for finite values of $\tilde r$ and $\tilde z \to \infty$. The decrease from $h = 1$ to $h = 0.5$ occurs at $\tilde r = -\infty$, $\tilde z = \infty$ and the further decrease to $h = 0$ at $\tilde r = \infty$, $\tilde z = \infty$.

Figure 4 shows lines where the azimuthal velocity g is constant. For $\tilde r < 0$ g is almost only dependent on $\tilde z$. This changes for $\tilde r > 0$ since then g increases to 1 for all $\tilde z$. For $\tilde z = 0$ there is an overshoot to about 1.11.

Lines of constant vorticity have been given in Fig. 5. All lines ω = constant are tangent to the semi-axis $\tilde r > 0$, $\tilde z = 0$. Approaching the origin in any other direction leads to $\omega \to \infty$.

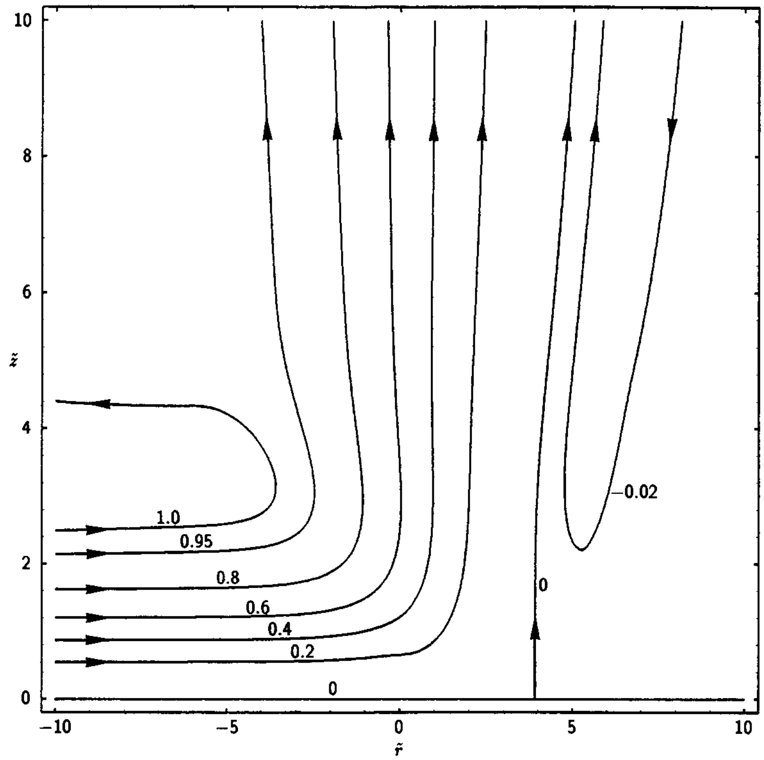

Fig. 3. Streamlines in the $\tilde{r}$, $\tilde{z}$-plane.

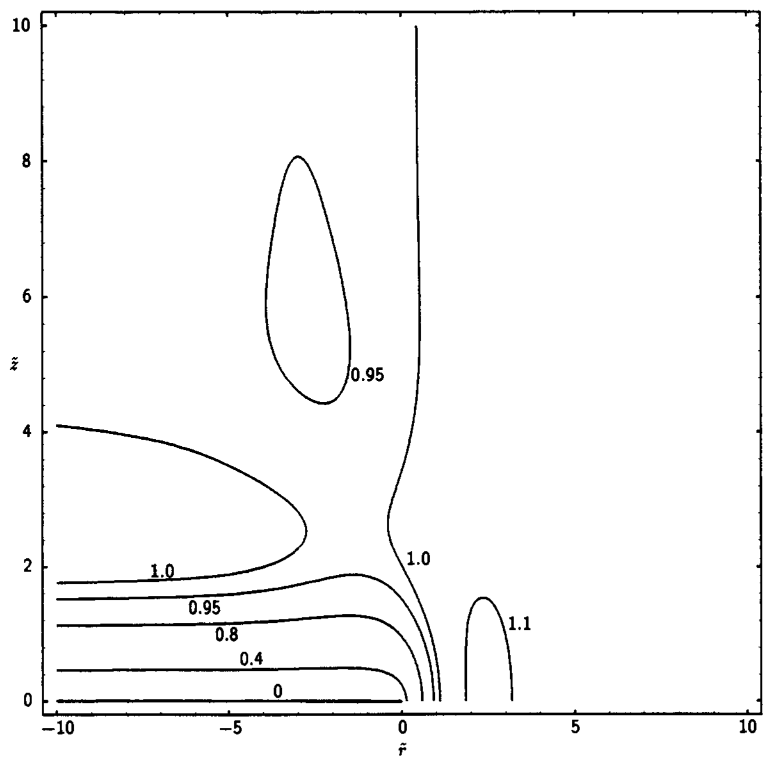

Fig. 4. The azimuthal velocity g in the $\tilde{r}$, $\tilde{z}$-plane.

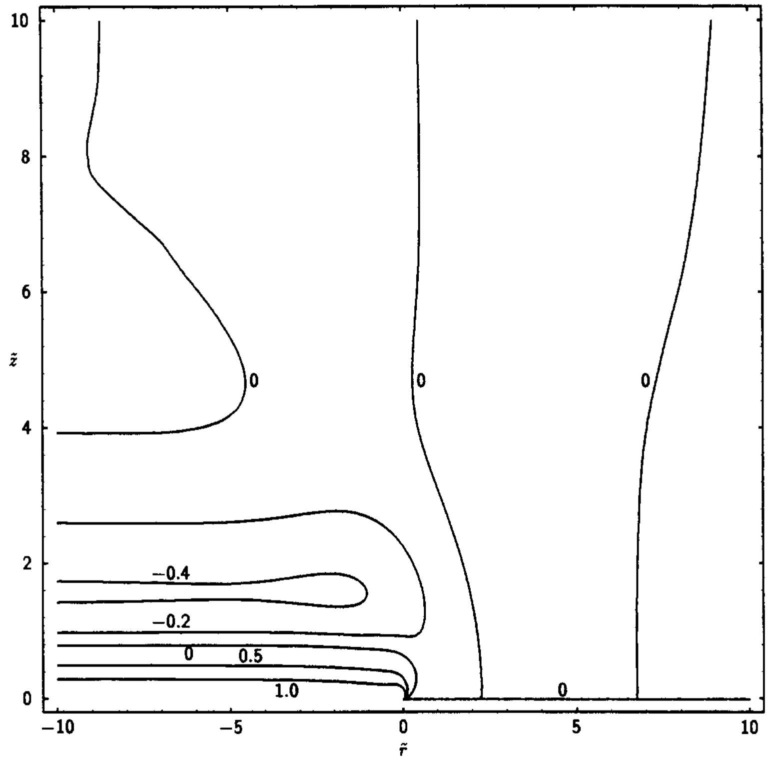

Fig. 5. The vorticity ω in the $\tilde{r}$, $\tilde{z}$-plane.

Figure 6 shows the radial velocity u and the azimuthal velocity g along the $\tilde{z}$-axis for $\tilde{r} > 0$. For $\tilde{r} > \sim 4$ u assumes small negative values due to the recirculating flow. The figure also shows the overshoot in g.

Finally, Fig. 7 presents the tangential shear stress $\partial g / \partial \tilde{z}$ and the vorticity, both along the disk. For $\tilde{r} \uparrow 0$ both functions behave like $\tilde{r}^{-1/2}$ and hence are integrable near the origin.

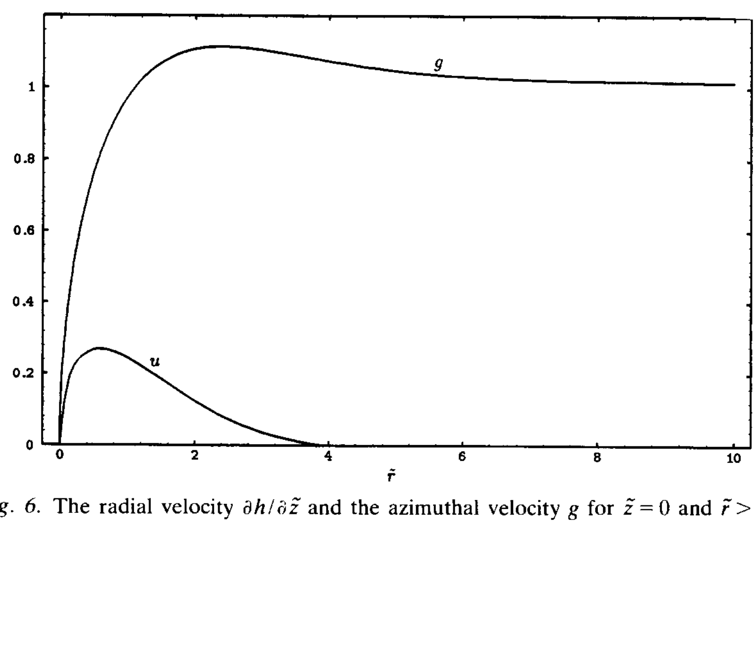

Fig. 6. The radial velocity $\partial h / \partial \tilde{z}$ and the azimuthal velocity g for $\tilde{z} = 0$ and $\tilde{r} > 0$.

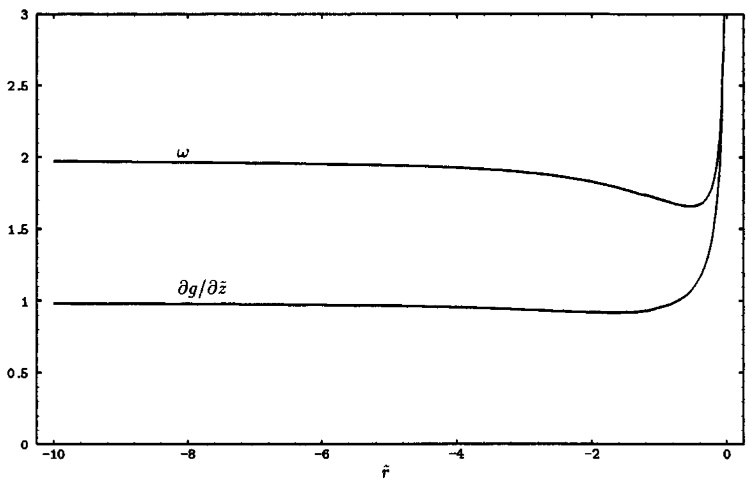

Fig. 7. The tangential shear stress $\partial g/\partial \tilde{z}$ and the vorticity ω for $\tilde{z}=0$ and $\tilde{r}<0$.

9. Conclusion

The flow in the region connecting the Ekman and Stewartson layers has been calculated by matching it to the flow in the two separate layers. The various properties of the flow are illustrated in Figs 3 to 7 and have been discussed in section 8 (Results).

Due to the decrease of the streamfunction as $O(\tilde{z}^{-1/3})$ to its final value at $\tilde{z}\to\infty$, it follows that the expansion of the streamfunction in the Stewartson layer for $E\to 0$ is

$$\psi = \varepsilon(\mathrm{E}^{1/2}\psi_1 + \mathrm{E}^{2/3}\psi_{3/2} + \mathrm{E}^{5/6}\psi_2 + \dots)$$

In the original paper [1] this expansion was given as

$$\psi = \varepsilon(\mathrm{E}^{1/2}\psi_1 + \mathrm{E}^{5/6}\psi_2 + \dots).$$

The solution ψ_1 of [1] remains unaltered, but $\psi_{3/2}$ is added and ψ_2 is slightly modified. The new solutions in the Stewartson layer satisfy homogeneous equations and homogeneous boundary conditions but arise from the singular point $r_1 = 0$, $z = 0$.

Acknowledgments

The author mentions gratefully discussions with E.F.F. Botta and A.E.P. Veldman and thanks K. Visser for producing the figures of this paper.

References

1. A.I. van de Vooren, The Stewartson layer of a rotating disk of finite radius. *J. Eng. Math.* 26 (1992) 131–152.
2. K. Stewartson, On almost rigid rotations. *J. Fluid Mech.* 3 (1957) 17–26.
3. K. Stewartson, On almost rigid rotations. Part 2. *J. Fluid Mech.* 26 (1966) 131–144.

4. D.W. Moore and P.G. Saffman, The rise of a body through a rotating fluid in a container of finite length. *J. Fluid Mech.* 31 (1968) 635–642.
5. D.W. Moore and P.G. Saffman, The flow induced by the transverse motion of a thin disk in its own plane through a contained rapidly rotating viscous liquid. *J. Fluid Mech.* 39 (1969) 831–847.
6. D.W. Moore and P.G. Saffman, The structure of free vertical shear layers in a rotating fluid and the motion produced by a slowly rising body. *Phil. Trans. Roy. Soc.* A 264 (1969) 597–634.
7. H.P. Greenspan, *The Theory of Rotating Flows*. Cambridge University Press, Cambridge (1968).
8. F.T. Smith, A note on a wall jet negotiating a trailing edge. *Q.J. Mech. Appl. Math.* 31 (1978) 473–479.
9. A.I. van de Vooren and E.F.F. Botta, The torque required for a steady rotation of a disk in a quiescent fluid. *J. Eng. Math.* 24 (1990) 261–286.